Stochastic models on a ring and quadratic algebras. The three-species diffusion problem

1998 J. Phys. A: Math. Gen. 31 833

(http://iopscience.iop.org/0305-4470/31/3/003)

J. Phys. A: Math. Gen. **31** (1998) 833–843. Printed in the UK PII: S0305-4470(98)84473-9

Stochastic models on a ring and quadratic algebras. The three-species diffusion problem

Peter F Arndt†, Thomas Heinzel† and Vladimir Rittenberg†
SISSA, Via Beirut 2–4, 34014 Trieste, Italy

Received 22 May 1997, in final form 15 July 1997

Abstract. The stationary state of a stochastic process on a ring can be expressed using traces of monomials of an associative algebra defined by quadratic relations. We consider only exclusion processes and restrict the type of algebras such that one has recurrence relations for traces of words of different lengths. This is possible only if the rates satisfy certain compatibility conditions. These conditions are derived and explicit representations of the generators of the quadratic algebras are given.

1. Introduction

In a previous paper [1] we considered the application of quadratic algebras to stochastic problems with closed or open boundaries. Here we study the case of periodic boundary conditions. We are again interested in the (unnormalized) probability distributions describing stationary states. In the language of quantum chains, we seek ground states which have zero momentum and energy. Much work has already been done in seeking matrix-product states in the case of periodic chains [2–4]. In the language of [1] in these papers polynomial algebras which have a trace operation were used. (Polynomial algebras are quadratic algebras without linear terms in the generators). In the present paper we consider a more general case in which the ground-state wavefunction can be written using quadratic algebras with linear terms. The idea of this approach is not new [5, 6]; all we have done is to pursue it in a consistent way in the case of models with three states. As a result one finds more solutions than were known previously, which can be used either to repeat the previous applications [5, 6] in a more general framework, or to look for novel applications. The same approach can be extended to problems with more states. Such an extension is not trivial.

From a mathematical point of view one has to solve a well stated problem: given a certain class of quadratic algebras one has to find those which are compatible with the trace operation. One lesson to be learned from the present work is that, unexpectedly, in order to solve the periodic case one makes use of Fock representations, derived in the previous paper [1], where the matrix product ansatz was applied to solve the problem with closed or open boundaries.

We first consider the general case of N species on a ring with L sites and use the notation of [1]. On each site we take a stochastic variable β_k ($\beta = 0, 1, \ldots, N-1$ and

† Permanent address: Physikalisches Institut, Nussallee 12, 53115 Bonn, Germany.

0305-4470/98/030833+11$19.50

$k = 1, 2, \ldots, L$), on each link k between the sites k and $k+1$ the rates $\Gamma^{\gamma_k \gamma_{k+1}}_{\beta_k \beta_{k+1}}$ give the probability per unit time for the transition:

$$\{\ldots, \gamma_k, \gamma_{k+1}, \ldots\} \mapsto \{\ldots, \beta_k, \beta_{k+1}, \ldots\}. \tag{1.1}$$

The Hamiltonian associated with the master equation [1] is

$$H = -\sum_{k=1}^{L} \Gamma^{\alpha\beta}_{\gamma\delta}\, E^{\gamma\alpha}_{k}\, E^{\delta\beta}_{k+1} \tag{1.2}$$

where the matrices E_k act on the kth site and have matrix elements

$$\left(E^{\alpha\beta}\right)_{\gamma\delta} = \delta_{\alpha\gamma}\delta_{\beta\delta} \tag{1.3}$$

and the diagonal elements $\Gamma^{\alpha\beta}_{\alpha\beta}$ are given by

$$\sum_{(\gamma,\delta)} \Gamma^{\alpha\beta}_{\gamma\delta} = 0. \tag{1.4}$$

The site $L+1$ is identified with the first site.

Now it is trivial to show that if we take N matrices D_α ($\alpha = 0, 1, \ldots, N-1$) and N matrices X_α satisfying the quadratic algebra

$$\sum_{\alpha,\beta=0}^{N-1} \Gamma^{\alpha\beta}_{\gamma\delta}\, D_\alpha\, D_\beta = X_\gamma\, D_\delta - D_\gamma\, X_\delta \qquad (\gamma, \delta = 0, 1, \ldots, N-1) \tag{1.5}$$

then

$$P_s = \mathrm{Tr}\left(\prod_{k=1}^{L}\left(\sum_{k=0}^{N-1} D_{\alpha_k} u^{(k)}_{\alpha_k}\right)\right) \tag{1.6}$$

is a stationary state:

$$H \cdot P_s = 0. \tag{1.7}$$

We have denoted by $u^{(k)}_{\alpha}$ ($\alpha = 0, 1, \ldots, N-1$ and $k = 1, 2, \ldots, L$) the basis vectors in the vector space associated with the kth site on which the basis matrices $E^{\alpha,\beta}_{k}$ of equation (1.2) act. The trace operation in (1.6) is taken in the auxiliary space of the D_α and X_α matrices. We note that the bulk algebra (1.5) is identical to that encountered in the previous paper. What is new here is the appearance of the trace operation in (1.6).

Let us observe that if one is interested only in one-dimensional representations of the algebra (1.5) (the matrices D are c-numbers), instead of the matrices X one takes c-numbers x, and one is left with a simple system of quadratic equations. In order to have solutions one obtains constraints on the rates gamma (this is a simple way to obtain, for example, the results of [7]; for other examples, see section 2).

In contrast to the problem with closed and open boundaries where the bulk algebra was completed by the condition of the existence of a Fock representation defined by the boundary conditions and where it was shown that D and X matrices can be derived once the bulk and boundary rates are given [8], in the present case very little is known, except that the bulk algebra exists, since a representation for the D's and X's is known [8]. This representation, however, is pathological in that the traces of any monomial of D's vanish. It is also not clear if all stationary states can be obtained through the ansatz (1.5).

The remarkable thing about the algebras (1.5) is that, if the ground state (1.6) is unique, all the traces of monomials of degree L and containing only D_α's and no X_α's are, up to a common factor, independent of the representation of the algebra, which implies that in order to compute them one can take the smallest one.

Last but not least, let us observe that the cases (see for example [4]) where the matrix product ansatz was applied correspond to representations with $X_\alpha = 0$. This leads us to polynomial algebras like those in [1, section 3].

We will now restrict our problem by looking only at simple exclusion processes. This means that only the rates $\Gamma^{\alpha\beta}_{\beta\alpha} = g_{\alpha\beta}$ and the diagonal ones are non-zero. Also, we will seek solutions in which the X_α matrices are c-numbers x_α. This last assumption will imply conditions on the rates $g_{\alpha\beta}$. The quadratic algebras now have a simple form:

$$g_{\alpha\beta}\, D_\alpha\, D_\beta - g_{\beta\alpha}\, D_\beta\, D_\alpha = x_\beta\, D_\alpha - x_\alpha\, D_\beta \qquad (\alpha, \beta = 0, 1, \ldots, N-1). \tag{1.8}$$

There are $N(N-1)/2$ relations with N parameters x_α and N generators D_α.

The appearance of N arbitrary parameters in the algebra can be understood in the following way. The problem has a $U(1)^{N-1}$ symmetry corresponding to the conservation of the number of particles of $N-1$ species (the remaining species are the vacancies). The ground state is highly degenerate. If one has a ring with L sites, P_s given by (1.6) has the following formal expression:

$$P_s = \sum_{n_\alpha} {d_0}^{n_0} {d_1}^{n_1} \cdots {d_{N-1}}^{n_{N-1}}\, A_{n_0, n_1, \ldots, n_{N-1}} \tag{1.9}$$

where

$$\sum_{\alpha=0}^{N-1} n_\alpha = L. \tag{1.10}$$

The d_α are arbitrary parameters and $A_{n_0, n_1, \ldots, n_{N-1}}$ are vectors on which H acts. In this way, in each sector, the ground state can be identified in terms of the number n_0 of vacancies and n_i of particles of type i. That P_s is indeed of the form (1.9) can be seen from the invariance of the algebra (1.8) under the transformation

$$D_\alpha \mapsto d_\alpha\, D_\alpha \qquad x_\alpha \mapsto d_\alpha\, x_\alpha. \tag{1.11}$$

Let us stress once again that in order to obtain the expression (1.9) one can take any representation of the algebra. We now turn our attention to the algebra (1.8).

What we need is not only to have associative algebras but also for the trace operation to exist. Since the left-hand side of equation (1.8) is quadratic and the right-hand side is linear in the generators D_α, this implies recurrence relations between traces of monomials and hence compatibility relations between the rates $g_{\alpha\beta}$. In order to solve this problem, our strategy is as follows: we first consider monomials of a low degree (up to five), obtain the compatibility relations and make sure that no new relations occur from higher-degree monomials by finding a representation with finite traces for the algebra. Once we have the algebra, one can carry out calculations for physical applications using either the algebra directly followed by formal manipulations under the trace operation, or use the representation. This procedure will be explained in detail in section 3 for the three-state case. In section 2 we ask a simple question: what are the conditions on the $g_{\alpha\beta}$ for arbitrary N such that one has one-dimensional representations? (One-dimensional representations obviously have a trace.) Finally, in section 4 we present our conclusions.

2. One-dimensional representations

The simplest examples of the algebras defined by equation (1.8) which have a trace are those in which one has one-dimensional representations. In order to find them, we take $D_\alpha = d_\alpha$, arbitrary non-zero c-numbers. It is convenient to introduce the notation

$$a_{\alpha\beta} = g_{\alpha\beta} - g_{\beta\alpha}. \tag{2.1}$$

Using equation (1.8) one obtains

$$\frac{x_\alpha}{d_\alpha} - \frac{x_\beta}{d_\beta} = a_{\beta\alpha} \qquad (\alpha, \beta = 0, 1, \ldots, N-1). \tag{2.2}$$

These equations determine the parameters x_α once the d_α are chosen. One obtains $(N-1)(N-2)/2$ conditions on the rates:

$$a_{0\alpha} - a_{0\beta} = a_{\beta\alpha} \qquad (\alpha, \beta = 1, 2, \ldots, N-1) \tag{2.3}$$

and the parameters x_α are

$$x_\alpha = d_\alpha (a_{0\alpha} + x_0/d_0) \qquad (\alpha = 1, 2, \ldots, N-1). \tag{2.4}$$

Note that one of the parameters can be chosen at will. In equations (2.3) and (2.4) we have singled out $\alpha = 0$ as a matter of notational convenience.

The wavefunction (see equation (1.6)) is symmetric. This observation is interesting for the following reason. In the case of simple exclusion processes, the Hamiltonian given by equation (1.2) has a $U(1)^{N-1}$ symmetry. As already discussed, this corresponds to conservation of the $N-1$ types of particles hopping between vacancies. The symmetric wavefunction, however, corresponds to an $SU(N)$ representation given by the Young tableau with one row and L boxes (L is the number of sites), although the Hamiltonian does not have this symmetry. From equation (2.3) we also see that for $N = 2$ one has one-dimensional representations for any rates.

3. The three-state algebras

We will now study the algebras given by equation (1.8) for $N = 3$ in detail. In order to find them, we will consider several cases.

3.1. x_0, x_1 and x_2 non-zero

We define

$$D_i = x_i\, E_i \qquad (i = 1, 2, 3) \tag{3.1}$$

and obtain the algebra

$$\begin{aligned} g_{01}\, E_0\, E_1 - g_{10}\, E_1\, E_0 &= E_0 - E_1 \\ g_{20}\, E_2\, E_0 - g_{02}\, E_0\, E_2 &= E_2 - E_0 \\ g_{12}\, E_1\, E_2 - g_{21}\, E_2\, E_1 &= E_1 - E_2. \end{aligned} \tag{3.2}$$

From writing the recurrence relations for monomials of degree two and three, the equations giving $\mathrm{Tr}(E_0\, E_1\, E_2)$ and $\mathrm{Tr}(E_2\, E_1\, E_0)$ are consistent only if

$$a_{01} - a_{02} = a_{21} \tag{3.3}$$

or if

$$\begin{aligned} &a_{12}\,(g_{10}\, g_{20} - g_{01}\, g_{02})\,\mathrm{Tr}(E_0) + a_{20}\,(g_{01}\, g_{21} - g_{10}\, g_{12})\,\mathrm{Tr}(E_1) \\ &\qquad + a_{01}\,(g_{02}\, g_{12} - g_{20}\, g_{21})\,\mathrm{Tr}(E_2) = 0. \end{aligned} \tag{3.4}$$

Equation (3.3) gives the condition for having a one-dimensional representation, see equation (2.3). Equation (3.4), however, is new. We use equation (3.4) to express $\mathrm{Tr}(E_0)$ in terms of $\mathrm{Tr}(E_1)$ and $\mathrm{Tr}(E_2)$ and look at the equations for monomials of degree four. No new conditions appear. (This implies that the ground states for chains of up to four sites can be obtained by this method.) For monomials of degree five, however, the consistency conditions for positive rates give the result that the traces of all monomials of degree two to four are zero. Since we went up to monomials of degree five one can guess what kind of 'dirty' algebra was required (see also [9]). We looked without success for conditions on the rates in order to find non-zero solutions. However, the equations are so cumbersome that we cannot even be sure that we did not miss one.

We have sought finite-dimensional representations of the algebra insisting on the positivity of the rates. In this way one can check part of the results obtained in an independent way by looking at the recurrence relations (the existence of a finite-dimensional representation implies supplementary conditions on the rates). We have looked at representations of dimensions two, three and four. The result was negative. If, however, we look at the purely algebraic problem of having an algebra with traces, we find, for example, that the algebra (3.2) exists if

$$g_{10}g_{01} = g_{20}g_{02} \qquad g_{12} = -g_{21} = \frac{g_{20}(g_{02} - g_{01})}{g_{01} + g_{20}}. \tag{3.5}$$

Note that this condition is incompatible with positivity of the rates. This algebra has a two-dimensional representation:

$$\begin{aligned} \mathcal{E}_0 &= \frac{1}{g_{20} - g_{01}} \begin{pmatrix} g_{01}/g_{02} & 0 \\ 0 & 1 \end{pmatrix} \\ \mathcal{E}_1 &= \frac{1}{g_{01} - g_{02}} \begin{pmatrix} 1 & 0 \\ -({g_{01}}^2 + {g_{20}}^2)/\lambda {g_{20}}^2 & g_{01}/g_{20} \end{pmatrix} \\ \mathcal{E}_2 &= \frac{1}{g_{02} - g_{01}} \begin{pmatrix} g_{01}/g_{20} & \lambda \\ 0 & 1 \end{pmatrix}. \end{aligned} \tag{3.6}$$

Here λ is an arbitrary parameter.

3.2. $x_0 = 0$, x_1 and x_2 non-zero

We define

$$D_1 = x_1 E_1 \qquad D_2 = x_2 E_2 \tag{3.7}$$

and the algebra (1.8) becomes

$$\begin{aligned} &g_{01}\, D_0\, E_1 - g_{10}\, E_1\, D_0 = D_0 \\ &g_{02}\, D_0\, E_2 - g_{20}\, E_2\, D_0 = D_0 \\ &g_{12}\, E_1\, E_2 - g_{21}\, E_2\, E_1 = E_1 - E_2. \end{aligned} \tag{3.8}$$

This algebra has a special structure in the sense that all the independent monomials in D_0, E_0 and E_2 can be organized in the following way:

$$P_0, \quad D_0 P_1, \quad {D_0}^2 P_2, \quad \ldots \tag{3.9}$$

where the P_i are monomials in E_1 and E_2 alone. In the trace problem this will imply a decoupling of the recurrence relations according to the power of D_0 appearing in the monomials. In particular, for words without D_0's, we can take $D_0 = 0$ in (3.8) and are left with the $N = 2$ algebra containing E_1 and E_2 for which we know that we have one-dimensional representations, and thus in this sector the problem is solved. The problem is of course to marry the last equation in (3.8) with the first two. The decoupling of the trace problem in various sectors will also have an unexpected consequence in the representations of the algebra. The representations with a trace for words containing D_0's will not have a trace for words not containing any D_0. So much for the structure of the algebra (3.8).

Going up to monomials of order three, the consistency relations obtained from the equations giving $\mathrm{Tr}(D_0\,E_1\,E_2)$ and $\mathrm{Tr}(E_2\,E_1\,D_0)$ again give equation (3.3) (this is compatible with (2.4) in which one can take $x_0 = 0$) or

$$g_{01}\,g_{02} = g_{10}\,g_{20}. \tag{3.10}$$

We first assume

$$g_{01},\ g_{20},\ g_{10},\ g_{02} \neq 0 \tag{3.11}$$

and look at words of order four. One obtains two new conditions which, together with equation (3.10), give

$$g_{10} = g_{02} \qquad g_{01} = g_{20} \qquad g_{21} - g_{12} = g_{01} - g_{10}. \tag{3.12}$$

We will now introduce the following notation:

$$q = \frac{g_{01}}{g_{10}} = \frac{g_{20}}{g_{02}} \qquad r = \frac{g_{21}}{g_{12}} \tag{3.13}$$

and

$$E_1 = \frac{G_1}{g_{01} - g_{10}} \qquad E_2 = \frac{G_2}{g_{10} - g_{01}}. \tag{3.14}$$

Taking into account the conditions (3.11) on the rates, the new algebra is

$$\begin{aligned} &q\,D_0\,G_1 - G_1\,D_0 = (q-1)\,D_0 \\ &q\,G_2\,D_0 - D_0\,G_2 = (q-1)\,D_0 \\ &r\,G_2\,G_1 - G_1\,G_2 = (r-1)\,(G_1 + G_2). \end{aligned} \tag{3.15}$$

At this point we are not going to look at words of order five or more, but will show that a representation with a trace exits. Before we show this, let us first note that the algebra (3.15) is invariant under the transformation

$$D_0 \mapsto D_0 \qquad G_1 \mapsto G_2 \qquad G_2 \mapsto G_1 \qquad q \mapsto \frac{1}{q} \qquad r \mapsto \frac{1}{r} \tag{3.16}$$

and that one has the identities

$$\begin{aligned} &q^n\,{D_0}^n\,G_1 - G_1\,{D_0}^n = (q^n - 1)\,{D_0}^n \\ &q^n\,G_2\,{D_0}^n - {D_0}^n\,G_2 = (q^n - 1)\,{D_0}^n. \end{aligned} \tag{3.17}$$

Equations (3.16) and (3.17) allow us to find traces on some monomials if one knows some others.

In order to show that a representation exists, we first write

$$\begin{aligned} &G_1 = 1 - \sqrt{r-1}\,\mathcal{A} \\ &G_2 = 1 + \sqrt{r-1}\,\mathcal{B} \\ &D_0 = d_0(1 + (q-1)\,\mathcal{N}) \end{aligned} \tag{3.18}$$

where d_0 is an arbitrary parameter. Using equation (3.15) we obtain

$$\begin{aligned} &\mathcal{AB} - r\mathcal{BA} = 1 \\ &\mathcal{AN} - q\mathcal{NA} = \mathcal{A} \\ &\mathcal{NB} - q\mathcal{BN} = \mathcal{B}. \end{aligned} \tag{3.19}$$

The algebra (3.19) which contains an r-deformed harmonic oscillator (with generators $\mathcal{A}$ and $\mathcal{B}$) together with q-deformed actions of the number operator $\mathcal{N}$ has a Fock representation [1, 10]:

$$\mathcal{A}|0\rangle = \langle 0|\mathcal{B} = 0 \qquad \mathcal{B} = \mathcal{A}^{\mathrm{T}} \tag{3.20}$$

where $\mathcal{A}^{\mathrm{T}}$ is the transpose of $\mathcal{A}$:

$$\mathcal{A} = \begin{pmatrix} 0 & g_1 & 0 & 0 & \cdots \\ 0 & 0 & g_2 & 0 & \\ 0 & 0 & 0 & g_3 & \\ \vdots & & & \ddots & \ddots \end{pmatrix} \qquad \mathcal{N} = \begin{pmatrix} p_1 & 0 & 0 & \cdots \\ 0 & p_2 & 0 & \\ 0 & 0 & p_3 & \\ \vdots & & & \ddots \end{pmatrix} \tag{3.21}$$

and

$$g_n{}^2 = \{n\}_r \qquad p_n = \{n-1\}_q \qquad \{n\}_\lambda = \frac{\lambda^n - 1}{\lambda - 1}. \tag{3.22}$$

It is convenient to denote

$$G_1 = 1 + \mathcal{F}_1 \qquad G_2 = 1 + \mathcal{F}_2 \qquad D_0 = d_0\, \mathcal{I}(q). \tag{3.23}$$

The matrices $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{I}(q)$ have a simple form:

$$\mathcal{F}_1 = \begin{pmatrix} 0 & -f_1 & 0 & 0 & \cdots \\ 0 & 0 & -f_2 & 0 & \\ 0 & 0 & 0 & -f_3 & \\ \vdots & & & \ddots & \ddots \end{pmatrix} \qquad \mathcal{F}_2 = -\mathcal{F}_1^{\mathrm{T}} \qquad \mathcal{I}(q) = \begin{pmatrix} e_1 & 0 & 0 & \cdots \\ 0 & e_2 & 0 & \\ 0 & 0 & e_3 & \\ \vdots & & & \ddots \end{pmatrix} \tag{3.24}$$

where

$$f_k{}^2 = r^k - 1 \qquad e_k = q^{k-1}. \tag{3.25}$$

Let us now note the following useful relations:

$$\begin{aligned} q\mathcal{I}(q)\mathcal{F}_1 &= \mathcal{F}_1\mathcal{I}(q) \\ q\mathcal{F}_2\mathcal{I}(q) &= \mathcal{I}(q)\mathcal{F}_2 \end{aligned} \tag{3.26}$$

and

$$\begin{aligned} \mathcal{F}_1\,\mathcal{F}_2 &= 1 - r\mathcal{I}(r) \\ \mathcal{F}_2\,\mathcal{F}_1 &= 1 - \mathcal{I}(r) \end{aligned} \tag{3.27}$$

as well as

$$\mathcal{I}(r)\mathcal{I}(q) = \mathcal{I}(rq) \tag{3.28}$$

and

$$\mathrm{Tr}\,\mathcal{I}(\lambda) = \frac{1}{1-\lambda}. \tag{3.29}$$

The calculation of the trace of any monomial containing at least one D_0 proceeds as follows. Using equation (3.24) one has to compute only traces containing an equal number of the $\mathcal{F}_1$'s and $\mathcal{F}_2$'s together with the $\mathcal{I}(q)$'s. We use the commutation relations (3.26) to bring the $\mathcal{I}(q)$'s together and 'condense' them into one using (3.28). Next we use equation (3.27) in order to express products of the $\mathcal{F}_1$'s and $\mathcal{F}_2$'s in terms of $\mathcal{I}(r)$'s and make them 'condense' together with the $\mathcal{I}(q^n)$ obtained from, let us say, n $\mathcal{I}(q)$'s. The final result is an expression which contains $\mathcal{I}$'s of various arguments, each one having a trace given by (3.29). The d_i of (1.9) are d_0, $d_1 = x_1$ and $d_2 = x_2$. This concludes our discussion of the algebra with the conditions (3.12). The special case $q = r$ is already known and was applied in [5].

We now return to (3.10) and consider the case

$$g_{01} = g_{20} = 0. \tag{3.30}$$

We can find directly representations of the algebra in this case.

First we consider

$$\mu \equiv \frac{g_{10}}{g_{21} - g_{12}} \neq -1 \qquad \nu \equiv \frac{g_{02}}{g_{21} - g_{12}} \neq -1. \tag{3.31}$$

We make a change of notation:

$$E_1 = -\frac{G_1}{g_{10}} \qquad E_2 = \frac{G_2}{g_{02}} \tag{3.32}$$

and instead of the algebra (3.8) we obtain

$$\begin{aligned} &G_1 D_0 = D_0 \\ &D_0 G_2 = D_0 \\ &g_{12} G_1 G_2 - g_{21} G_2 G_1 = g_{02} G_1 + g_{10} G_2. \end{aligned} \tag{3.33}$$

Similar to what we did in the case of the previous algebra (see equation (3.23)), we write

$$G_1 = 1 + \mathcal{F}_1 \qquad G_2 = 1 + \mathcal{F}_2 \qquad D_0 = d_0 \mathcal{F}_0 \tag{3.34}$$

and make the observation that the Fock representation of the following algebra [1, 10]:

$$\begin{aligned} &\mathcal{A}\mathcal{I}(0) = 0 \\ &\mathcal{I}(0)\mathcal{B} = 0 \\ &\xi(\mathcal{A}\mathcal{B} - r\,\mathcal{B}\mathcal{A}) = \mathcal{A} + \mathcal{B} + 1 \\ &\mathcal{A}|0\rangle = \langle 0|\mathcal{B} = 0 \\ &\mathcal{B} = \mathcal{A}^{\mathrm{T}} \end{aligned} \tag{3.35}$$

is known

$$\mathcal{A} = \begin{pmatrix} a_1 & k_1 & 0 & 0 & \cdots \\ 0 & a_2 & k_2 & 0 & \\ 0 & 0 & a_3 & k_3 & \\ \vdots & & & \ddots & \ddots \end{pmatrix} \tag{3.36}$$

where

$$a_n = \frac{1}{\xi}\,\frac{r^{n-1} - 1}{r - 1} \qquad {k_n}^2 = \frac{1}{\xi^2}\,\frac{r^n - 1}{r - 1}\left(\xi + \frac{r^{n-1} - 1}{r - 1}\right) \tag{3.37}$$

and $\mathcal{I}(q = 0)$ as in (3.24). We have introduced the following notation for the ratios of the rates:

$$r = \frac{g_{21}}{g_{12}} \qquad \xi = \frac{\mu + \nu + 1}{(\mu + 1)(\nu + 1)}. \tag{3.38}$$

Using equations (3.35)–(3.37) we find a representation for $\mathcal{F}_0$, $\mathcal{F}_1$ and $\mathcal{F}_2$ of (3.34):

$$\begin{aligned}\mathcal{F}_0 &= \mathcal{I}(0)\\ \mathcal{F}_1 &= (\mu+1)(\mathcal{I}(r)+\mathcal{V}-1)\\ \mathcal{F}_2 &= (\nu+1)(\mathcal{I}(r)+\mathcal{V}^{\mathrm{T}}-1)\end{aligned} \tag{3.39}$$

where $\mathcal{I}(r)$ is defined as above and

$$\mathcal{V} = \begin{pmatrix} 0 & s_1 & 0 & 0 & \cdots \\ 0 & 0 & s_2 & 0 & \\ 0 & 0 & 0 & s_3 & \\ \vdots & & & \ddots & \ddots \end{pmatrix} \tag{3.40}$$

with

$$s_n{}^2 = (r^n - 1)(\xi + r^{n-1} - 1). \tag{3.41}$$

Note that

$$r\mathcal{I}(r)\mathcal{V} = \mathcal{V}\mathcal{I}(r) \tag{3.42}$$

and that the products $\mathcal{V}\mathcal{V}^{\mathrm{T}}$ and $\mathcal{V}^{\mathrm{T}}\mathcal{V}$ can be written in terms of $\mathcal{I}(r)$ and $\mathcal{I}(r^2)$. This makes the discussion of the existence and calculations of the traces identical to that for the previous algebra. A special case of this algebra when $r = 0$ has already been discussed in [5, 6].

For the case $\mu = -1$ one must use a different representation (the case $\nu = -1$ is similar). We write

$$E_1 = \frac{1}{g_{10}}\left(1 + \frac{g_{02}}{g_{02}-g_{10}}\mathcal{A}\right) \qquad E_2 = -\frac{1}{g_{02}}\left(1 + \frac{g_{02}-g_{10}}{g_{12}}\mathcal{B}\right) \qquad D_0 = d_0\,\mathcal{I}(0) \tag{3.43}$$

where $\mathcal{A}$ and $\mathcal{B}$ now fulfill

$$\begin{aligned}&\mathcal{A}\mathcal{I}(0) = 0\\ &\mathcal{I}(0)\mathcal{B} = 0\\ &\mathcal{A}\mathcal{B} - r\mathcal{B}\mathcal{A} = \mathcal{A} + 1\\ &\mathcal{A}|0\rangle = \langle 0|\mathcal{B} = 0.\end{aligned} \tag{3.44}$$

The matrices are given by

$$\mathcal{A} = \begin{pmatrix} 0 & f_1 & 0 & 0 & \cdots \\ 0 & 0 & f_2 & 0 & \\ 0 & 0 & 0 & f_3 & \\ 0 & 0 & 0 & 0 & \ddots \\ \vdots & & & & \ddots \end{pmatrix} \qquad \mathcal{B} = \begin{pmatrix} b_1 & 0 & 0 & 0 & \cdots \\ f_1 & b_2 & 0 & 0 & \\ 0 & f_2 & b_3 & 0 & \\ 0 & 0 & f_3 & b_4 & \\ \vdots & & & \ddots & \ddots \end{pmatrix} \tag{3.45}$$

with

$$b_n = \frac{r^{n-1}-1}{r-1} \qquad f_n{}^2 = \frac{r^n - 1}{r-1} \tag{3.46}$$

(cf equations (3.35)–(3.37)).

The profound reason which explains why the same representations for the E_α occur in the problem with periodic boundary conditions and for the open-ended problems (see

[1, section 7]) comes from the following observation. The differences between the two types of boundary conditions is that the parameters x_1 and x_2 are free for the periodic case but are determined by the boundary matrices in the open case. In the latter case supplementary relations exist between the bulk ($g_{\alpha\beta}$) and boundary rates. Since the E_α found in the case of open ends have a trace, they can be used in the present paper.

3.3. $x_0 = x_1 = 0$, x_2 non-zero

In this case the algebra (1.8) becomes

$$\begin{aligned} &g_{02}\, D_0\, E_2 - g_{20}\, E_2\, D_0 = D_0 \\ &g_{12}\, D_1\, E_2 - g_{21}\, E_2\, D_1 = D_1 \\ &g_{01}\, D_0\, D_1 - g_{10}\, D_1\, D_0 = 0 \end{aligned} \tag{3.47}$$

where we have taken $D_2 = x_2\, E_2$. If one wants non-trivial ground states with particles of the three species, one has to take

$$g_{01} = g_{10} = 0. \tag{3.48}$$

In this case the symmetry of the problem is enormous, since no prescription is given in terms of products of D_0's and D_1's. The right language comprises affine symmetries [11] and it is beyond the scope of this paper to get involved in the representation theory for this case. The algebra with traces exists and its representations are known [12]. It has already been used in [13] in the sector of one D_0 and one D_1. This model was also investigated in [14] by different methods.

Finally, the case $x_0 = x_1 = x_2 = 0$ brings us to the situation of symmetric rates where, as we already know, we obtain a symmetric wavefunction.

4. Conclusion

For the three-species diffusion problem we have shown (see section 3) that if and only if the rates $g_{\alpha\beta}$ satisfy one of the conditions (3.3), (3.12), (3.30) or (3.48), the algebra (1.8) has a representation with a finite trace. For the cases (3.3), (3.12), and (3.30) we give these representations. For the case (3.48) the representation is also known [12]. Knowing these representations and using (1.6) one can determine various correlation functions in the stationary state. For a long chain this can be a very tedious calculation.

Leaving aside physical applications, our own fascination with the problem described in this paper comes from the unusual properties of the representations of the algebra appearing in searching for matrix-product ground states. As already stressed in the introduction, certain monomials or even all monomials of the same degree in the D's have, up to a normalization factor, traces independent of the representation. More has to be understood in the general case when the X's are matrices (see equation (1.5)) and also in the class of algebras given by equation (1.8). It is probably possible to encode the conditions on the rates $g_{\alpha\beta}$ in some cubic and quartic identities. This guess is based on the fact that conditions found on the $g_{\alpha\beta}$ were obtained from words of degree three and four. For this reason the four-state problem is worth looking at in order to see if a general pattern appears.

Acknowledgments

We would like to thank SISSA and our colleagues here for the warm and stimulating environment and the EU and DAAD for financial support. We are grateful to

S Dasmahapatra, B Derrida, B Dubrovin, M Evans, A Honecker, K Mallick and P Martin for discussions.

References

[1] Alcaraz F C, Dasmahapatra S and Rittenberg V 1997 N-species stochastic models with boundaries and quadratic algebras *Preprint* cond-mat/9705172
[2] Hakim V and Nadal J P 1983 *J. Phys. A: Math. Gen.* **16** L213
[3] Fannes M, Nachtergale B and Werner R F 1996 *Commun. Math. Phys.* **123** 477 and references therein
[4] Niggemann H, Klümper A and Zittarz J 1997 *Preprint* cond-mat/9702178 and references therein
[5] Derrida B, Janowsky S A, Lebowitz J L and Speer E R 1993 *J. Stat. Phys.* **71** 813
[6] Mallick K 1996 *J. Phys. A: Math. Gen.* **29** 5375
[7] Neergard J and M den Nijs 1997 *J. Phys. A: Math. Gen.* **30** 1935
[8] Krebs K and Sandow S 1997 *J. Phys. A: Math. Gen.* to be published
[9] Honecker A 1997 private communication
[10] Essler F H L and Rittenberg V 1996 *J. Phys. A: Math. Gen.* **29** 3375
[11] Alcaraz F C, Arnaudon D, Rittenberg V and Scheunert M 1994 *Int. J. Mod. Phys.* A **9** 3473 and references therein
[12] Evans M R 1997 private communication
[13] Evans M R 1996 *Europhys. Lett.* **36** 13
[14] Menon G I, Barma M and Dhar D 1997 *J. Stat. Phys.* **86** 1237